Solution to Problem Set 2
Optical Waveguides and Fibers (OWF)

Exercise 1: Dispersion of fused silica (amorphous SiO_2)

The refractive indices of dielectric materials can be written in functional form by means of the Sellmeier equations. For fused silica, the most common material used for fabricating optical fibers, the Sellmeier equation takes the following form:

$$n^2(\lambda) = 1 + \frac{0.6962\lambda^2}{\lambda^2 - (0.06840)^2} + \frac{0.4079\lambda^2}{\lambda^2 - (0.1162)^2} + \frac{0.8975\lambda^2}{\lambda^2 - (9.8962)^2}. \quad (1)$$

The quantity λ denotes the vacuum wavelength in micrometers.

a) Generate a computer plot, e.g., using MATLAB, that shows the refractive index of fused silica as a function of wavelength. Eq. (1) is valid between $0.2\,\mu$m and $3.7\,\mu$m, i.e., from the ultraviolet region to the near infrared.

Hint: MATLAB can be accessed from any Computer at the SCC. For home use, a licence can be downloaded by any student via the SCC: http://www.scc.kit.edu/produkte/3841.php.

Solution: Using Eq. (1) and $n(\lambda) = \sqrt{n^2(\lambda)}$ one gets the plot in Fig. 1a

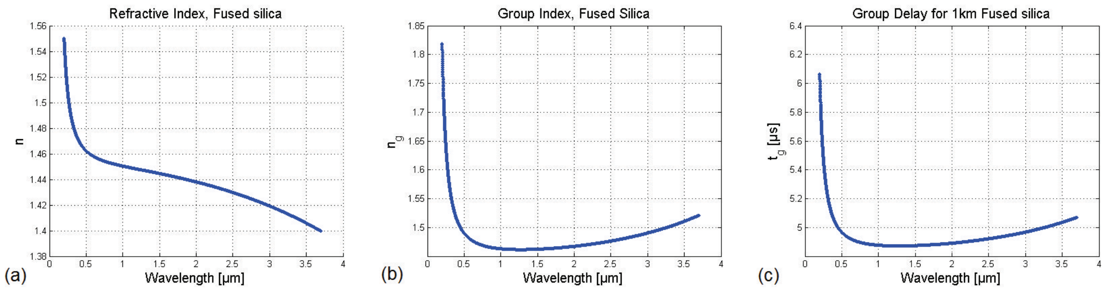

Figure 1: Bulk refractive index and group index in silica, calculated with the Sellmeier equation.

b) Consider a pulse of light with center wavelength λ propagating over 1 km through bulk fused silica. Plot the arrival time as a function of λ.

Solution: The propagation time of a pulse is defined by the group velocity $v_g = \frac{\partial\omega}{\partial k} = \frac{c}{n_g}$, with the group refractive index $n_g = n(\lambda) - \lambda\frac{\partial n}{\partial \lambda}$. The derivation can be approximated in the numerical plot with the difference quotient, the group refractive index is plotted in Fig. 1b. With that the group delay $t_g = \frac{z}{v_g} = \frac{z}{c}n_g$ of a pulse in $L = 1$km bulk silica can be calculated, the plot can be seen in Fig. 1c.

c) Short pulses have broad spectra, i.e., they consist of various different wavelength components. Which center wavelength would you choose to transmit a short pulse through bulk fused silica with minimum impairment?

Solution: From the calculation of the group delay we see that different frequencies have different propagation times. Assuming a short pulse with a broad spectrum we want that all frequency components see the same group delay to maintain the pulse shape. In the plot of the group delay we see that this is the case around the minimum of the group delay at $1.27\,\mu$m.

d) Plot the material dispersion coefficient M_λ as a function of wavelength on a scale having the units $\frac{\text{ps}}{\text{km nm}}$, which are the most common units for this quantity.

Solution: The material dispersion is defined as $M_\lambda = \frac{1}{c}\frac{\partial n_g}{\partial \lambda}$, with the difference quotients the differential of the group index can be numerically calculated. The plot of the material dispersion can be seen in Fig. 2, the point where the material dispersion is zero is also the point, where a pulse propagating through bulk material has the least impairment which is at the wavelength of $\lambda = 1.27\text{µm}$.

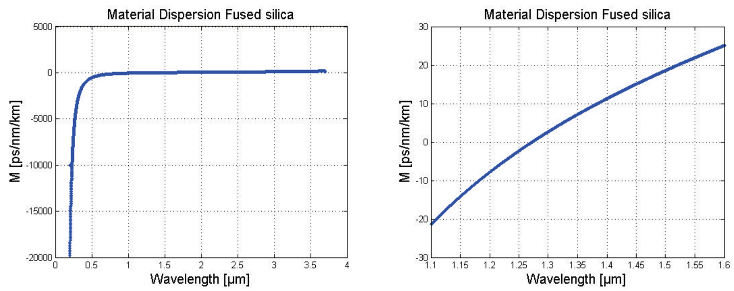

Figure 2: Material dispersion of silica

Exercise 2: Spreading of a Gaussian pulse as it propagates in a dispersive medium.

Consider a pulse which is propagating along the z direction within a material having material dispersion M_λ at the carrier angular frequency ω_c. Assume that at $z = 0$ the pulse is described by:

$$\underline{a}(z=0,t) = \underline{A}_0 e^{-\frac{t^2}{2\sigma_t^2(0)}} e^{\mathrm{j}\omega_c t} \tag{2}$$

a) Calculate $\underline{a}(z,t)$ for $z > 0$. To do so, you can proceed in the following way:

- Calculate the Fourier transform of the pulse.
- Assume a complex propagator of the form $e^{-\mathrm{j}\beta(\omega)z}$. Use a Taylor expansion up to second order to approximate the propagation constant, i.e., $\beta(\omega) = \beta_c + \beta_c^{(1)}(\omega-\omega_c) + \frac{1}{2}\beta_c^{(2)}(\omega-\omega_c)^2$.
- Perform the inverse Fourier transform. Hint: Introduce the quantity $\underline{\sigma_t^2}(z) = \sigma_t^2(0) + \mathrm{j}\beta_c^{(2)}z$

Solution: Fourier Transform:

$$\mathcal{F}\left[\underline{a}(z=0,t)\right] = \underline{\tilde{a}}(z=0,\omega) = \underline{A}_0\sqrt{2\pi}\sigma_t(0)e^{-\frac{\sigma_t^2(0)(\omega-\omega_c)^2}{2}} \tag{3}$$

Multiplication with complex propagator and Taylor expansion up to second order:

$$\underline{\tilde{a}}(z,\omega) = \underline{A}_0\sqrt{2\pi}\sigma_t(0)e^{-\frac{\sigma_t^2(0)(\omega-\omega_c)^2}{2}}e^{-\mathrm{j}\beta(\omega)z} \tag{4}$$

$$\simeq \underline{A}_0\sqrt{2\pi}\sigma_t(0)e^{-\frac{\sigma_t^2(0)(\omega-\omega_c)^2}{2}}e^{-\mathrm{j}\left(\beta_c^{(0)} + \beta_c^{(1)}(\omega-\omega_c) + \frac{1}{2}\beta_c^{(2)}(\omega-\omega_c)^2\right)z} \tag{5}$$

Using the relation $\underline{\sigma_t^2}(z) = \sigma_t^2(0) + \mathrm{j}\beta_c^{(2)}z$, and $\delta\omega = \omega - \omega_c$ we can write

$$\underline{\tilde{a}}(z,\omega) = \underline{A}_0\frac{\sigma_t(0)}{\underline{\sigma_t}(z)}e^{-\mathrm{j}\beta_c z}\sqrt{2\pi}\underline{\sigma_t}(z)e^{-\frac{\underline{\sigma_t^2}(z)\delta\omega^2}{2}}e^{-\mathrm{j}\beta_c^{(1)}\delta\omega z} \tag{6}$$

To perform the inverse Fourier transform $\underline{a}(z,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\underline{\tilde{a}}(z,\omega)e^{\mathrm{j}\omega t}\mathrm{d}\omega$ we write

$$\underline{a}(z,t) = e^{\mathrm{j}\omega_c t}\underline{A}_0\frac{\sigma_t(0)}{\underline{\sigma_t}(z)}e^{-\mathrm{j}\beta_c z}\frac{\sqrt{2\pi}}{2\pi}\underline{\sigma_t}(z)\int e^{-\frac{\underline{\sigma_t^2}(z)\delta\omega^2}{2}}e^{-\mathrm{j}\beta_c^{(1)}\delta\omega z}e^{\mathrm{j}\omega t}e^{-\mathrm{j}\omega_c t}\mathrm{d}\omega \tag{7}$$

$$= e^{\mathrm{j}\omega_c t}\underline{A}_0\frac{\sigma_t(0)}{\underline{\sigma_t}(z)}e^{-\mathrm{j}\beta_c z}\frac{\sqrt{2\pi}}{2\pi}\underline{\sigma_t}(z)\int e^{-\frac{\underline{\sigma_t^2}(z)\delta\omega^2}{2}}e^{-\mathrm{j}\beta_c^{(1)}\delta\omega z}e^{\mathrm{j}\delta\omega t}\mathrm{d}\delta\omega \tag{8}$$

$$= e^{\mathrm{j}\omega_c t}\underline{A}_0\frac{\sigma_t(0)}{\underline{\sigma_t}(z)}e^{-\mathrm{j}\beta_c z}\frac{\sqrt{2\pi}}{2\pi}\underline{\sigma_t}(z)\int e^{-\frac{\underline{\sigma_t^2}(z)\delta\omega^2}{2}}e^{-\mathrm{j}\delta\omega(t-\beta_c^{(1)}z)}\mathrm{d}\delta\omega \tag{9}$$

Now we can make use of hint 2 to transform the Gaussian pulse and also we see that $e^{-\mathrm{j}\beta_c^{(1)}\delta\omega}$ transforms into a time shift $(t-\beta_c^{(1)}z)$ after the Fourier transform.

$$\underline{a}(z,t) = \underline{A}_0 \frac{\sigma_t(0)}{\underline{\sigma}_t(z)} e^{\mathrm{j}(\omega_c t-\beta_c z)} e^{-\frac{(t-\beta_c^{(1)}z)^2}{2\underline{\sigma}_t^2(z)}} \tag{10}$$

b) Show that the pulse remains Gaussian and that

$$|\underline{a}(z,t)| \propto e^{-\frac{\left(t-\beta_c^{(1)}z\right)^2}{2\sigma_t^2(z)}}, \tag{11}$$

where

$$\sigma_t^2(z) = \sigma_t^2(0) + \frac{\left(\beta_c^{(2)}z\right)^2}{\sigma_t^2(0)}. \tag{12}$$

Solution: In the Eq. (10) $\underline{\sigma}_t^2(z)$ is still a complex number, if we now use again $\underline{\sigma}_t^2(z) = \sigma_t^2(0) + \mathrm{j}\beta_c^{(2)}z$ we can write

$$\frac{1}{\underline{\sigma}_t^2(z)} = \frac{1}{\sigma_t^2(0)+\mathrm{j}\beta_c^{(2)}z} \cdot \frac{\sigma_t^2(0)-\mathrm{j}\beta_c^{(2)}z}{\sigma_t^2(0)-\mathrm{j}\beta_c^{(2)}z} \tag{13}$$

$$= \frac{\sigma_t^2(0)-\mathrm{j}\beta_c^{(2)}z}{\sigma_t^4(0)+(\beta_c^{(2)}z)^2} \tag{14}$$

$$= \frac{1}{\sigma_t^2(0)+\left(\frac{\beta_c^{(2)}z}{\sigma_t(0)}\right)^2} - \mathrm{j}\frac{\beta_c^{(2)}z}{\sigma_t^2(0)\left(\sigma_t^2(0)+\left(\frac{\beta_c^{(2)}z}{\sigma_t(0)}\right)^2\right)} = \frac{1}{\sigma_t^2(z)} - \mathrm{j}\frac{\beta_c^{(2)}z}{\sigma_t^2(0)\left(\sigma_t^2(z)\right)} \tag{15}$$

by introducing the real variable $\sigma_t^2(z) = \sigma_t^2(0) + \left(\frac{\beta_c^{(2)}z}{\sigma_t(0)}\right)^2$. We can insert this now into the previously obtained equation for $\underline{a}(z,t)$ and get

$$\underline{a}(z,t) = \underline{A}_0 \frac{\sigma_t(0)}{\underline{\sigma}_t(z)} e^{\mathrm{j}\frac{\beta_c^{(2)}z}{2\sigma_t^2(0)\sigma_t^2(z)}(t-\beta_c^{(1)}z)^2} \cdot e^{-\frac{(t-\beta_c^{(1)}z)^2}{\sigma_t^2(z)}} \cdot e^{\mathrm{j}(\omega_c t-\beta_c^{(0)}z)} \tag{16}$$

c) How do $\beta_c^{(0)}$, $\beta_c^{(1)}$ and $\beta_c^{(2)}$ influence the optical signal?

Solution:

In Eq. (16) the term $e^{-\mathrm{j}\beta_c^{(0)}z}$ is adding a constant phase shift, depending on z and $\beta_c^{(0)}$.

$\beta_c^{(1)}z$ corresponds to a time shift, $t_g = \beta_c^{(1)}z = \frac{z}{v_g} = \frac{zn_g}{c}$

In the parameter $\sigma_t^2(z) = \sigma_t^2(0) + \left(\frac{\beta_c^{(2)}z}{\sigma_t(0)}\right)^2$ the value of $\beta_c^{(2)}$ broadens the Gaussian pulse after propagation. Furthermore it adds a frequency chirp onto the signal as it can be seen from Eq. (16).

Note 1: Fourier transform convention

Remember that in this course we use the following definition of the Fourier transform.

$$\mathcal{F}\left[\underline{f}(t)\right] = \underline{\tilde{f}}(\omega) = \int_{-\infty}^{+\infty} \underline{f}(t)e^{-\mathrm{j}\omega t}\mathrm{d}t \tag{17}$$

The latter equation implies that the inverse transform is:

$$\mathcal{F}^{-1}\left[\underline{\tilde{f}}(\omega)\right] = \underline{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underline{\tilde{f}}(\omega)e^{\mathrm{j}\omega t}\mathrm{d}\omega \tag{18}$$

Note 2: Fourier transform of the Gaussian function

According to the previous definition:

$$\mathcal{F}\left[e^{-\frac{t^2}{2\sigma^2}}\right] = \sqrt{2\pi}\sigma e^{-\frac{\sigma^2\omega^2}{2}}, \text{ for Re}\left[\frac{1}{\sigma^2} > 0\right]. \tag{19}$$

Note 3: Relation between the Taylor expansion of $\beta(\omega)$ and the material dispersion M_λ

The definition of the material dispersion coefficient M_λ is:

$$M_\lambda = \frac{\Delta t_g}{z\Delta\lambda}. \tag{20}$$

This relation can be easily remembered when recalling the previously introduced dimension, $\frac{\text{ps}}{\text{km nm}}$: M_λ gives the group delay spread Δt_g in ps between two wavepackets for which the center wavelengths are separated by $\Delta\lambda = 1\text{nm}$ after a propagation distance of $z = 1\text{km}$.
A useful relation between M_λ and $\beta_c^{(2)}$ can be obtained when using

$$t_g \equiv \frac{z}{v_g} = z\beta_c^{(1)}, \tag{21}$$

in Eq. (20)

$$M_\lambda = \frac{\mathrm{d}\beta_c^{(1)}}{\mathrm{d}\lambda} = -\frac{\omega}{\lambda}\frac{\mathrm{d}\beta_c^{(1)}}{\mathrm{d}\omega} = -\frac{2\pi c}{\lambda^2}\beta_c^{(2)}. \tag{22}$$

Questions and Comments:

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